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Algebraic Bethe ansatz for 19-vertex models with reflection conditions

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Abstract
In this work we solve the 19-vertex models with the use of algebraic Bethe ansatz for diagonal reflection matrices (Sklyanin *K*-matrices). The eigenvectors, eigenvalues and Bethe equations are given in a general form. Quantum spin chains of spin one derived from the 19-vertex models were also discussed.

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1. Introduction

Classical statistical systems in two spatial dimensions on a lattice (vertex models) and one-dimensional quantum spin chain Hamiltonians share a common mathematical structure responsible for our understanding of these integrable models [1–3]. If the Boltzmann weights underlying the vertex models are obtained from solutions of the Yang–Baxter (YB) equation the commutativity of the associated transfer matrices immediately follows, leading to their integrability.

The diagonalizations of these models can be made with the use of the Bethe ansatz (BA). It is a powerful method in the analysis of integrable quantum models. There are several versions: coordinate BA [4], algebraic BA [5], analytical BA [6], etc.

The algebraic BA, also known as the quantum inverse scattering method, is based on the idea of constructing eigenfunctions of the Hamiltonian via creation and annihilation operators acting on a reference state. Here one uses the fact that the YB equation can be recast in the form of commutation relations for the matrix elements of the monodromy matrix which play the role of creation and annihilation operators. From this monodromy matrix we get the transfer matrix which commutes with the Hamiltonian.

Imposing appropriate boundary conditions the BA method leads to a system of equations, the Bethe equations, which are useful in the thermodynamic limit. The energy of the ground state and its excitations, velocity of sound, etc, may be calculated in this limit. Moreover, in recent years we witnessed another very fruitful connection between the BA method and conformal field theory. Using the algebraic BA, Korepin [7] found various representations of correlators in integrable models and more recently Babujian and Flume [8] developed a method from the algebraic BA which reveals a link to the Gaudin model, rendering solutions of the Knizhnik–Zamolodchikov equations for the $SU(2)$ Wess–Zumino–Novikov–Witten conformal theory in the quasi-classical limit.

Integrable quantum systems containing Fermi fields have been attracting increasing interest due to their potential applications in condensed matter physics. The prototypical examples of such systems are the supersymmetric generalizations of the Hubbard and t–J models [9]. They lead to a generalization of the YB equation associated with the introduction of a Z_2 grading [10] which leads to the appearance of additional signs in the YB equation.

When considering systems on a finite interval with independent boundary conditions at each end, we have to introduce reflection matrices to describe such boundary conditions. Integrable models with boundaries can be constructed out of a pair of reflection K-matrices $K^{\pm}(u)$ in addition to the solution of the YB equation. Here $K^{-}(u)$ and $K^{+}(u)$ describe the effects of the presence of boundaries at the left and right ends, respectively.

Integrability of open chains in the framework of the quantum inverse scattering method was pioneered by Sklyanin relying on previous results of Cherednik [11]. In [12], Sklyanin used his formalism to solve, via algebraic BA, the open spin-$1/2$ chain with diagonal boundary terms. This model was already solved via coordinate BA by Alcaraz *et al* [13]. The Sklyanin original formalism was extended to more general systems by Mezincescu and Nepomechie in [14]. Doikou in [43] using the fusion technique and the analytical Bethe ansatz computed the K-matrix and solved the $A_{N-1}^{(1)}$-spin chain models. More recently in [44] the classification of the rational K-matrices was extended for the series $so(m)$, $sp(n)$ and $osp(m|n)$ models.

In this paper we consider the algebraic version of the BA for the three-state vertex models with a class of boundary terms derived from diagonal reflection K-matrices. These models are well known in the literature: the Zamolodchikov–Fateev (ZF) model or $A_1^{(1)}$ model [15], the Izergin–Korepin (IK) model or $A_2^{(2)}$ model [16] and two Z_2-graded models, named the $sl(2|1)$ model and the $osp(1|2)$ model [17].

In [27], Fan used the algebraic Bethe ansatz to study the IK model with a specific $K(u)$-diagonal reflection matrix. The main goal in this paper is to generalize the work of Fan to all 19-vertex models and for diagonal reflection matrices.

We introduce the algebraic tools in section 2 where we define the models to be studied and the diagonal reflection matrices related to them. In section 3, we apply the algebraic BA method with periodic boundary condition to all models present in section 2. Many identities and relations that appear in the case of a periodic boundary condition will be useful in the next step. In section 4, we study the case with reflection conditions and the energy spectra and the corresponding Bethe equations are presented in a general form. In section 5 we studied some models derived from the 19-vertex models. These spin-1 quantum chains were classified by Idzumi *et al* in [32]. The conclusions are in section 6. In appendix A we discuss the model IK with the $R(u)$ presented in the original paper of Fan [27] and show that the results present in [27] are reproduced as a special case for our formulation.

2. 19-vertex models and reflection matrices

Before we begin to study the algebraic Bethe ansatz with reflection condition, we will need to define the algebraic structure for the models that will be studied. In this section, we will briefly discuss these models and give the reflection matrices for them. A detailed discussion about the trigonometric reflection matrices for all 19-vertex models can be found in [30] (see also [44] for the classification of the rational K-matrices for $osp(m|n)$ models).

To determine an integrable vertex model on a lattice it is first necessary that the bulk vertex weights be specified by an $\mathcal{R}$-matrix $\mathcal{R}(u)$, where u is the spectral parameter. It acts on the tensor product $V^1 \otimes V^2$ for a given vector space V and satisfies a special system of functional equations, the YB equation

$$\mathcal{R}_{12}(u)\mathcal{R}_{13}(u+v)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(u+v)\mathcal{R}_{12}(u) \tag{1}$$

in $V^1 \otimes V^2 \otimes V^3$, where $\mathcal{R}_{12} = \mathcal{R} \otimes \mathbf{1}$, $\mathcal{R}_{23} = \mathbf{1} \otimes \mathcal{R}$, etc.

An $\mathcal{R}$ matrix is said to be regular if it satisfies the property $\mathcal{R}(0) = P$, where P is the permutation matrix in $V^1 \otimes V^2$: $P(|\alpha\rangle \otimes |\beta\rangle) = |\beta\rangle \otimes |\alpha\rangle$ for $|\alpha\rangle, |\beta\rangle \in V$. In addition, we will require [14] that $\mathcal{R}(u)$ satisfies the following properties

$$\begin{array}{lll} \text{regularity} & : & \mathcal{R}_{12}(0) = f(0)^{1/2} P_{12} \\ \text{unitarity} & : & \mathcal{R}_{12}(u)\mathcal{R}_{12}^{t_1 t_2}(-u) = f(u) \\ \text{PT-symmetry} & : & P_{12}\mathcal{R}_{12}(u)P_{12} = \mathcal{R}_{12}^{t_1 t_2}(u) \\ \text{crossing-symmetry} & : & \mathcal{R}_{12}(u) = U_1 \mathcal{R}_{12}^{t_2}(-u-\rho) U_1^{-1} \end{array} \tag{2}$$

where $f(u) = x_1(u)x_1(-u)$, t_i denotes transposition in the space i , ρ is the crossing parameter and U determines the crossing matrix

$$M = U^t U = M^t. \tag{3}$$

Note that unitarity and crossing-symmetry together imply the useful relation

$$M_1 \mathcal{R}_{12}^{t_2}(-u-\rho) M_1^{-1} \mathcal{R}_{12}^{t_1}(u-\rho) = f(u). \tag{4}$$

The boundary weights then follow from K-matrices which satisfy boundary versions of the YB equation [12, 14]: the reflection equation

$$\mathcal{R}_{12}(u-v)K_1^-(u)\mathcal{R}_{12}^{t_1 t_2}(u+v)K_2^-(v) = K_2^-(v)\mathcal{R}_{12}(u+v)K_1^-(u)\mathcal{R}_{12}^{t_1 t_2}(u-v) \tag{5}$$

and the dual reflection equation

$$\begin{aligned} &\mathcal{R}_{12}(-u+v)\left(K_1^+\right)^{t_1}(u)M_1^{-1}\mathcal{R}_{12}^{t_1 t_2}(-u-v-2\rho)M_1\left(K_2^+\right)^{t_2}(v) \\ &\quad = \left(K_2^+\right)^{t_2}(v)M_1\mathcal{R}_{12}(-u-v-2\rho)M_1^{-1}\left(K_1^+\right)^{t_1}(u)\mathcal{R}_{12}^{t_1 t_2}(-u+v). \end{aligned} \tag{6}$$

In this case there is an isomorphism between K^- and K^+:

$$K^-(u) :\to K^+(u) = K^-(-u-\rho)^t M. \tag{7}$$

Therefore, given a solution to the reflection equation (5) we can also find a solution to the dual reflection equation (6).

In the framework of the quantum inverse scattering method, we define the Lax operator from the $\mathcal{R}$-matrix as $\mathcal{L}_{\rm aq}(u) = \mathcal{R}_{\rm aq}(u)$, where the subscript 'a' represents auxiliary space, and 'q' represents quantum space. The row-to-row monodromy matrix $T(u)$ is defined as a matrix product over the N operators on all sites of the lattice,

$$T(u) = \mathcal{L}_{aN}(u)\mathcal{L}_{aN-1}(u)\cdots\mathcal{L}_{a1}(u). \tag{8}$$

The main result is the following: if the boundary equations are satisfied, then the Sklyanin transfer matrix

$$t(u) = \mathrm{Tr}_a(K^+(u)T(u)K^-(u)T^{-1}(-u)) \tag{9}$$

forms a commuting family

$$[t(u), t(v)] = 0 \qquad \forall u, v. \tag{10}$$

The commutativity of $t(u)$ can be proved by using the unitarity and crossing-unitarity relations, the reflection equation and the dual reflection equation. It implies integrability of an open quantum spin chain whose Hamiltonian (with $K^-(0) = 1$) can be obtained as

$$H = \sum_{k=1}^{N-1} H_{k,k+1} + \frac{1}{2} \left. \frac{\mathrm{d}K_1^-(u)}{\mathrm{d}u} \right|_{u=0} + \frac{\mathrm{tr}_0\, K_0^+(0) H_{N,0}}{\mathrm{tr}\, K^+(0)} \tag{11}$$

and whose two-site terms are given by

$$H_{k,k+1} = \left. \frac{\mathrm{d}}{\mathrm{d}u} P_{k,k+1} \mathcal{R}_{k,k+1}(u) \right|_{u=0} \tag{12}$$

in the standard fashion.

Here we will extend our discussions to include the Z_2-graded vertex models. Therefore, let us describe some useful information about the graded formulation.

Let $V = V_0 \oplus V_1$ be a Z_2-graded vector space where 0 and 1 denote the even and odd parts, respectively. Multiplication rules in the graded tensor product space $V \overset{s}{\otimes} V$ differ from the ordinary ones by the appearance of additional signs. The components of a linear operator $A \overset{s}{\otimes} B \in V \overset{s}{\otimes} V$ result in matrix elements of the form

$$(A \overset{s}{\otimes} B)_{\alpha\beta}^{\gamma\delta} = (-)^{p(\beta)(p(\alpha)+p(\gamma))} A_{\alpha\gamma} B_{\beta\delta}. \tag{13}$$

The action of the graded permutation operator $\mathcal{P}$ on the vector $|\alpha\rangle \overset{s}{\otimes} |\beta\rangle \in V \overset{s}{\otimes} V$ is defined by

$$\mathcal{P}|\alpha\rangle \overset{s}{\otimes} |\beta\rangle = (-)^{p(\alpha)p(\beta)} |\beta\rangle \overset{s}{\otimes} |\alpha\rangle \Longrightarrow (\mathcal{P})_{\alpha\beta}^{\gamma\delta} = (-)^{p(\alpha)p(\beta)} \delta_{\alpha\delta}\delta_{\beta\gamma}. \tag{14}$$

The graded transposition 'st' and the graded trace 'str' are defined by

$$(A^{\mathrm{st}})_{\alpha\beta} = (-)^{(p(\alpha)+1)p(\beta)} A_{\beta\alpha} \qquad \mathrm{str}\, A = \sum_{\alpha} (-)^{p(\alpha)} A_{\alpha\alpha} \tag{15}$$

where $p(\alpha) = 1$ (0) if $|\alpha\rangle$ is an odd (even) element.

For the graded case the YB equation and the reflection equation remain the same as above. We only need to change the usual tensor product to the graded tensor product.

In general, the dual reflection equation which depends on the unitarity and cross-unitarity relations of the $\mathcal{R}$-matrix takes different forms for different models. For the models considered in this paper, we write the graded dual reflection equation in the following form [18]:

$$\begin{aligned} &\mathcal{R}_{21}^{\mathrm{st}_1\mathrm{st}_2}(-u+v)\big(K_1^+\big)^{\mathrm{st}_1}(u) M_1^{-1} \mathcal{R}_{12}^{\mathrm{st}_1\mathrm{st}_2}(-u-v-2\rho) M_1 \big(K_2^+\big)^{\mathrm{st}_2}(v) \\ &\quad = \big(K_2^+\big)^{\mathrm{st}_2}(v) M_1 \mathcal{R}_{12}^{\mathrm{st}_1\mathrm{st}_2}(-u-v-2\rho) M_1^{-1} \big(K_1^+\big)^{\mathrm{st}_1}(u) \mathcal{R}_{21}^{\mathrm{st}_1\mathrm{st}_2}(-u+v) \end{aligned} \tag{16}$$

and will choose a common parity assignment: $p(1) = p(3) = 0$ and $p(2) = 1$, the BFB grading.

Now, using the relations

$$\mathcal{R}_{12}^{\mathrm{st}_1\mathrm{st}_2}(u) = I_1 R_{21}(u) I_1 \qquad \mathcal{R}_{21}^{\mathrm{st}_1\mathrm{st}_2}(u) = I_1 R_{12}(u) I_1 \qquad \text{and} \qquad I K^+(u) I = K^+(u) \tag{17}$$

with $I = \mathrm{diag}(1, -1, 1)$ and the property $[M_1 M_2, \mathcal{R}(u)] = 0$ we can see that the isomorphism (7) holds with the BFB grading.

The three-state vertex models that we will consider are the Zamolodchikov–Fateev (ZF) model, the Izergin–Korepin (IK) model, the $sl(2|1)$ model and the $osp(1|2)$ model. Their $\mathcal{R}$-matrices have a common form

$$\mathcal{R}(u) = \left(\begin{array}{ccc|ccc|ccc} x_1 & & & & & & & & \\ & x_2 & & x_5 & & & & & \\ & & x_3 & & x_6 & & x_7 & & \\ \hline & y_5 & & x_2 & & & & & \\ & & y_6 & & x_4 & & x_6 & & \\ & & & & & x_2 & & x_5 & \\ \hline & & y_7 & & y_6 & & x_3 & & \\ & & & & & y_5 & & x_2 & \\ & & & & & & & & x_1 \end{array}\right) \tag{18}$$

satisfying properties (1)–(4) together with their graded version.

2.1. The Zamolodchikov–Fateev model

The simplest three-state vertex model is the ZF 19-vertex [15] or the $A_1^{(1)}$ model the spin-1 representation [20] and can be constructed from the six-vertex model using the fusion procedure. The $\mathcal{R}$-matrix which satisfies the YB equation (1) has the form (18) with

$$\begin{aligned} &x_1(u) = \sinh(u+\eta)\sinh(u+2\eta) && x_2(u) = \sinh u \sinh(u+\eta) \\ &x_3(u) = \sinh u \sinh(u-\eta) && x_4(u) = \sinh u \sinh(u+\eta) + \sinh\eta \sinh 2\eta \\ &y_5(u) = x_5(u) = \sinh(u+\eta)\sinh 2\eta && y_6(u) = x_6(u) = \sinh u \sinh 2\eta \\ &y_7(u) = x_7(u) = \sinh\eta \sinh 2\eta. && \end{aligned} \tag{19}$$

This $\mathcal{R}$-matrix is regular and unitary, with $f(u) = x_1(u)x_1(-u)$, P- and T-symmetric and crossing-symmetric with $M = 1$ and $\rho = \eta$. The most general diagonal solution for $K^-(u)$ has been obtained in [19] and is given by

$$K^-(u, \beta_{11}) = \begin{pmatrix} k_{11}^-(u) & & \\ & 1 & \\ & & k_{33}^-(u) \end{pmatrix} \tag{20}$$

with

$$k_{11}^-(u) = -\frac{\beta_{11}\sinh u + 2\cosh u}{\beta_{11}\sinh u - 2\cosh u} \qquad k_{33}^-(u) = -\frac{\beta_{11}\sinh(u+\eta) - 2\cosh(u+\eta)}{\beta_{11}\sinh(u-\eta) + 2\cosh(u-\eta)} \tag{21}$$

where β_{11} is the free parameter. By automorphism (7) the solution for $K^+(u)$ follows

$$K^+(u, \alpha_{11}) = K^-(-u-\rho, \alpha_{11}) = \begin{pmatrix} k_{11}^+(u) & & \\ & 1 & \\ & & k_{33}^+(u) \end{pmatrix} \tag{22}$$

with

$$\begin{aligned} k_{11}^+(u) &= -\frac{\alpha_{11}\sinh(u+\eta) - 2\cosh(u+\eta)}{\alpha_{11}\sinh(u+\eta) + 2\cosh(u+\eta)} \\ k_{33}^+(u) &= -\frac{\alpha_{11}\sinh u + 2\cosh u}{\alpha_{11}\sinh(u+2\eta) - 2\cosh(u+2\eta)} \end{aligned} \tag{23}$$

where α_{11} is another free parameter.

For a particular choice of boundary terms, the ZF spin chain has the quantum group symmetry, i.e. if we choose $\xi_\mp \to \infty$ ($\beta_{11} = 2\coth\xi_-$ and $\alpha_{11} = 2\coth\xi_+$), then the spin chain Hamiltonian (19) has $U_q(su(2))$-invariance [19].

2.2. The Izergin–Korepin model

The solution of the YB equation corresponding to $A_2^{(2)}$ in the fundamental representation was found by Izergin and Korepin [16]. The $\mathcal{R}$-matrix has the form (18) with non-zero entries

$$\begin{aligned}
&x_1(u) = \sinh(u-5\eta) + \sinh\eta \qquad x_2(u) = \sinh(u-3\eta) + \sinh 3\eta \\
&x_3(u) = \sinh(u-\eta) + \sinh\eta \qquad x_4(u) = \sinh(u-3\eta) - \sinh 5\eta + \sinh 3\eta + \sinh\eta \\
&x_5(u) = -2\,\mathrm{e}^{-u/2}\sinh 2\eta\cosh\left(\frac{u}{2}-3\eta\right) \qquad y_5(u) = -2\,\mathrm{e}^{u/2}\sinh 2\eta\cosh\left(\frac{u}{2}-3\eta\right) \\
&x_6(u) = 2\,\mathrm{e}^{-u/2+2\eta}\sinh 2\eta\sinh\left(\frac{u}{2}\right) \qquad y_6(u) = -2\,\mathrm{e}^{u/2-2\eta}\sinh 2\eta\sinh\left(\frac{u}{2}\right) \\
&x_7(u) = -2\,\mathrm{e}^{-u+2\eta}\sinh\eta\sinh 2\eta - \mathrm{e}^{-\eta}\sinh 4\eta \\
&y_7(u) = 2\,\mathrm{e}^{u-2\eta}\sinh\eta\sinh 2\eta - \mathrm{e}^{\eta}\sinh 4\eta.
\end{aligned} \tag{24}$$

This $\mathcal{R}$-matrix is regular and unitary, with $f(u) = x_1(u)x_1(-u)$. It is PT-symmetric and crossing-symmetric, with $\rho = -6\eta - \mathrm{i}\pi$ and

$$M = \begin{pmatrix} \mathrm{e}^{2\eta} & & \\ & 1 & \\ & & \mathrm{e}^{-2\eta} \end{pmatrix}. \tag{25}$$

Diagonal solutions for $K^-(u)$ have been obtained in [22]. It turns out that there are three solutions without free parameters, being $K^-(u) = 1$, $K^-(u) = F^+$ and $K^-(u) = F^-$, with

$$F^{\pm} = \begin{pmatrix} \mathrm{e}^{-u} f^{(\pm)}(u) & & \\ & 1 & \\ & & \mathrm{e}^{u} f^{(\pm)}(u) \end{pmatrix} \tag{26}$$

where we have defined

$$f^{(\pm)}(u) = \frac{\cosh(u/2-3\eta) \pm \mathrm{i}\sinh(u/2)}{\cosh(u/2-3\eta) \mp \mathrm{i}\sinh(u/2)}. \tag{27}$$

By automorphism (7), three solutions $K^+(u)$ follow as $K^+(u) = M, K^+(u) = G^+$ and $K^+(u) = G^-$, with

$$G^{\pm} = \begin{pmatrix} \mathrm{e}^{u-4\eta} g^{(\pm)}(u) & & \\ & 1 & \\ & & \mathrm{e}^{-u+4\eta} g^{(\pm)}(u) \end{pmatrix} \tag{28}$$

where we have defined

$$g^{(\pm)}(u) = \frac{\cosh(u/2-3\eta) \pm \mathrm{i}\sinh(u/2)}{\cosh(u/2-3\eta) \mp \mathrm{i}\sinh(u/2-6\eta)}. \tag{29}$$

Finally, we note that it is interesting to reformulate the Boltzmann weights of the IK model by the following transformation:

$$\mathcal{R}(u,\eta) \to \mathcal{R}'(u,\eta) = \frac{1}{2\mathrm{i}}\mathcal{R}\left(2u, -\eta - \mathrm{i}\frac{\pi}{2}\right). \tag{30}$$

This $\mathcal{R}'$ matrix differs from that given in [26] by a gauge transformation. It is regular and unitary, with $f'(u) = x_1'(u)x_1'(-u)$, PT-symmetric and crossing-unitarity with $M' = \mathrm{diag}(-\mathrm{e}^{-2\eta}, 1, -\mathrm{e}^{2\eta})$ and $\rho' = 3\eta$. After the gauge transformation $\mathcal{R}''_{12}(u) = V_1\mathcal{R}'_{12}(u)V_1^{-1}$ with $V = \mathrm{diag}(\mathrm{e}^{-u}, 1, \mathrm{e}^{u})$, we can see that $M'' = \mathrm{diag}(-\mathrm{e}^{4\eta}, 1, -\mathrm{e}^{-4\eta})$ and $\rho'' = \rho'$. In this case the solution (F^+, G^+) can be written as

$$F''^{-} = \mathrm{diag}\left(1, -\frac{\sinh\left(u-\frac{3}{2}\eta\right)}{\sinh\left(u+\frac{3}{2}\eta\right)}, 1\right) \qquad G''^{+} = -\mathrm{diag}\left(\mathrm{e}^{4\eta}, \frac{\sinh\left(u+\frac{9}{2}\eta\right)}{\sinh\left(u+\frac{3}{2}\eta\right)}, \mathrm{e}^{-4\eta}\right). \tag{31}$$

This solution was used by Fan in [27] to find the spectrum of the corresponding transfer matrix using the algebraic BA for one- and two-particle excited states.

2.3. The $sl(2|1)$ model

The solution of the graded YB equation corresponding to $sl(2|1)$ in the fundamental representation has the form (18) with non-zero entries [17, 28]:

$$\begin{aligned}
&x_1(u) = \cosh(u+\eta)\sinh(u+2\eta) && x_2(u) = \sinh u \cosh(u+\eta) \\
&x_3(u) = \sinh u \cosh(u-\eta) && x_4(u) = \sinh u \cosh(u+\eta) - \sinh 2\eta \cosh\eta \\
&y_5(u) = x_5(u) = \sinh 2\eta \cosh(u+\eta) && y_6(u) = x_6(u) = \sinh 2\eta \sinh u \\
&y_7(u) = x_7(u) = \sinh 2\eta \cosh\eta.
\end{aligned} \tag{32}$$

This $\mathcal{R}$-matrix is regular and unitary, with $f(u) = x_1(u)x_1(-u)$, P- and T-symmetric and crossing-symmetric with $M = 1$ and $\rho = \eta$. The graded version of the crossing-unitarity relation (4) is satisfied with $f'(u) = x_1\big(u + \mathrm{i}\frac{\pi}{2}\big)x_1\big(-u - \mathrm{i}\frac{\pi}{2}\big)$.

The most general diagonal solution for $K^-(u)$ has been presented in [29] and is given by

$$K^-(u,\beta_{11}) = \begin{pmatrix} k_{11}^-(u) & & \\ & 1 & \\ & & k_{33}^-(u) \end{pmatrix} \tag{33}$$

with

$$k_{11}^-(u) = -\frac{\beta_{11}\sinh u + 2\cosh u}{\beta_{11}\sinh u - 2\cosh u} \qquad k_{33}^-(u) = \frac{\beta_{11}\cosh(u+\eta) - 2\sinh(u+\eta)}{\beta_{11}\cosh(u-\eta) + 2\sinh(u-\eta)} \tag{34}$$

where β_{11} is the free parameter. Due to automorphism (7) the solution for $K^+(u)$ is given by $K^-\big(-u-\rho, \frac{1}{4}\alpha_{11}\big)$, i.e.

$$K^+(u,\beta_{11}) = \begin{pmatrix} k_{11}^+(u) & & \\ & 1 & \\ & & k_{33}^+(u) \end{pmatrix} \tag{35}$$

where

$$\begin{aligned}
k_{11}^+(u) &= \frac{\alpha_{11}\cosh(u+\eta) - 2\sinh(u+\eta)}{\alpha_{11}\cosh(u+\eta) + 2\sinh(u+\eta)} \\
k_{33}^+(u) &= -\frac{\alpha_{11}\sinh u + 2\cosh u}{\alpha_{11}\sinh(u+2\eta) - 2\cosh(u+2\eta)}
\end{aligned} \tag{36}$$

and α_{11} is another free parameter.

2.4. The $osp(1|2)$ model

The trigonometric solution of the graded YB equation corresponding to $osp(1|2)$ in the fundamental representation has the form (18) with non-zero entries [17]:

$$\begin{aligned}
&x_1(u) = \sinh(u+2\eta)\sinh(u+3\eta) && x_2(u) = \sinh u \sinh(u+3\eta) \\
&x_3(u) = \sinh u \sinh(u+\eta) && x_4(u) = \sinh u \sinh(u+3\eta) - \sinh 2\eta \sinh 3\eta \\
&x_5(u) = \mathrm{e}^{-u}\sinh 2\eta \sinh(u+3\eta) && y_5(u) = \mathrm{e}^{u}\sinh 2\eta \sinh(u+3\eta) \\
&x_6(u) = -\mathrm{e}^{-u-2\eta}\sinh 2\eta \sinh u && y_6(u) = \mathrm{e}^{u+2\eta}\sinh 2\eta \sinh u \\
&x_7(u) = \mathrm{e}^{-u}\sinh 2\eta(\sinh(u+3\eta) + \mathrm{e}^{-\eta}\sinh u) \\
&y_7(u) = \mathrm{e}^{u}\sinh 2\eta(\sinh(u+3\eta) + \mathrm{e}^{\eta}\sinh u).
\end{aligned} \tag{37}$$

This $\mathcal{R}$-matrix is regular and unitary, with $f(u) = x_1(u)x_1(-u)$. It is PT-symmetric and crossing-symmetric, with $\rho = 3\eta$ and

$$M = \begin{pmatrix} e^{2\eta} & & \\ & 1 & \\ & & e^{-2\eta} \end{pmatrix}. \tag{38}$$

Diagonal solutions for $K^-(u)$ have been obtained in [30] and rational solutions were obtained in [44]. It turns out that there are three solutions without free parameters, being $K^-(u) = 1$, $K^-(u) = F^+$ and $K^-(u) = F^-$, with

$$F^\pm = \begin{pmatrix} \mp e^{-2u} f^{(\pm)}(u) & & \\ & 1 & \\ & & \mp e^{2u} f^{(\pm)}(u) \end{pmatrix} \tag{39}$$

where we have defined

$$f^{(+)}(u) = \frac{\sinh(u + 3\eta/2)}{\sinh(u - 3\eta/2)} \qquad f^{(-)}(u) = \frac{\cosh(u + 3\eta/2)}{\cosh(u - 3\eta/2)}. \tag{40}$$

By the automorphism (7), three solutions $K^+(u)$ follow as $K^+(u) = M$, $K^+(u) = G^+$ and $K^+(u) = G^-$, with

$$G^\pm = \begin{pmatrix} \mp e^{2u+4\eta} g^{(\pm)}(u) & & \\ & 1 & \\ & & \mp e^{-2u-4\eta} g^{(\pm)}(u) \end{pmatrix} \tag{41}$$

where we have defined

$$g^{(+)}(u) = \frac{\sinh(u + 3\eta/2)}{\sinh(u + 9\eta/2)} \qquad g^{(-)}(u) = \frac{\cosh(u + 3\eta/2)}{\cosh(u + 9\eta/2)}. \tag{42}$$

2.5. From non-graded to graded solutions

Besides the $\mathcal{R}$-matrix we also have the R-matrix, which satisfies

$$R_{12}(u)R_{23}(u+v)R_{12}(v) = R_{23}(v)R_{12}(u+v)R_{23}(u). \tag{43}$$

Because only R_{12} and R_{23} are involved, this equation written in components looks the same as in the non-graded case. Moreover, the matrix $\mathcal{R} = PR$ satisfies the usual YB equation (1) where P is the non-graded permutation matrix. When the graded permutation matrix $\mathcal{P}$ is used, then $\mathcal{R} = \mathcal{P}R$ satisfies the graded version of the YB equation.

Multiplying the $\mathcal{R}$-matrix for 19-vertex models (18) by the diagonal matrix $\Pi = P\mathcal{P} = \mathcal{P}P$ we will get graded $\mathcal{R}$-matrices starting from non-graded $\mathcal{R}$-matrices and vice versa. The new $\mathcal{R}$-matrix $\mathcal{R}' = \Pi\mathcal{R}$ still has the form (18) but with the change of sign of the fifth row due to the grading BFB. Now $\epsilon = -1$ for non-graded models and $\epsilon = 1$ for graded models.

Let us use this interchange property with the YB solution of the IK model. First we recall the transformation (30)

$$\mathcal{R}'(u, \eta) = \frac{1}{2i}\mathcal{R}\left(2u, -\eta - i\frac{\pi}{2}\right). \tag{44}$$

The matrix $\mathcal{R}_{IKg}(u, \eta) = \Pi\mathcal{R}'$ is a solution of the graded version of the YB equation (1) and the corresponding vertex model can be named as the graded version of the IK model.

Using the symmetries of the YB solutions for 19-vertex models: $x_2(u) \to \pm x_2(u)$ and $x_6(u) \to \pm x_6(u)$ with $y_6(u) \to \mp y_6(u)$, we can see that this model has the same Boltzmann weights as the $osp(1|2)$ model, except for the presence of the factor $\pm i$ in $x_6(u)$ and $\mp i$ in

$y_6(u)$. However, this identification is not so trivial due to the change in the signs of the fifth row of $\mathcal{R}$ (BFB grading). Nevertheless, by direct computation we have verified that both models have the same reflection K-matrices. It means that $\mathcal{R}_{IKg}(u,\eta)$ and the $\mathcal{R}(u,\eta)$ of the $osp(1|2)$ share the same symmetries.

This situation is also present in the graded version of the ZF model. In order to see that we have to reformulate conveniently the Boltzmann weights of the ZF model by the following transformation:

$$\mathcal{R}(u,\eta) \to \mathcal{R}'(u,\eta) = \frac{1}{\mathrm{i}}\mathcal{R}\left(u, \eta - \mathrm{i}\frac{\pi}{2}\right). \tag{45}$$

The graded version of the ZF model is defined by the following $\mathcal{R}$-matrix:

$$\mathcal{R}_{ZFg}(u,\eta) = \Pi\mathcal{R}'(u,\eta). \tag{46}$$

Using again the symmetries of the 19-vertex model we can see, up to a possible canonical transformation: $x_6 \to x_6'(u) = \pm \mathrm{i} x_6(u)$, the non-zero entries of $\mathcal{R}_{ZFg}(u,\eta)$ are identified with the Boltzmann weights of the $sl(2|1)$ model (32). We note that both models have the same K-matrices and their coordinate Bethe ansatz yield a common spectrum.

It is possible to note that the inverse situation is also true. The non-graded versions of the graded 19-vertex models are in correspondence with the 19-vertex models of Izergin–Korepin and Zamolodchikov–Fateev.

In [31] Saleur and Wehefritz-Kaufmann studied the connection between Izergin–Korepin and $osp(1|2)$ models.

3. The algebraic Bethe ansatz with periodic boundary condition

We will first give a brief review of the algebraic Bethe ansatz for the 19-vertex models with periodic boundary condition and in the following we will discuss the case with reflection condition.

The graded quantum inverse scattering method is characterized by the monodromy matrix $T(u)$ satisfying the equation

$$R(u-v)[T(u) \overset{s}{\otimes} T(v)] = [T(v) \overset{s}{\otimes} T(u)]R(u-v) \tag{47}$$

whose consistency is guaranteed by the graded version of the YB equation (43). $T(u)$ is a matrix in the space V (the auxiliary space) whose matrix elements are operators on the states of the quantum system (the quantum space, which will also be the space V). The monodromy operator $T(u)$ is defined as an ordered product of local operators $\mathcal{L}_n$ (Lax operator), on all sites of the lattice:

$$T(u) = \mathcal{L}_N(u)\mathcal{L}_{N-1}(u)\cdots\mathcal{L}_1(u). \tag{48}$$

The Lax operator on the nth quantum space can be written in the form

$$\mathcal{L}_n = \frac{1}{x_2}\begin{pmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & x_6 & 0 & x_7 & 0 & 0 \\ 0 & y_5 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_6 & 0 & x_4 & 0 & x_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_2 & 0 & x_5 & 0 \\ 0 & 0 & y_7 & 0 & y_6 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_5 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 \end{pmatrix} = \begin{pmatrix} L_{11}^{(n)}(u) & L_{12}^{(n)}(u) & L_{13}^{(n)}(u) \\ L_{21}^{(n)}(u) & L_{22}^{(n)}(u) & L_{23}^{(n)}(u) \\ L_{31}^{(n)}(u) & L_{32}^{(n)}(u) & L_{33}^{(n)}(u) \end{pmatrix}. \tag{49}$$

Note that $L^{(n)}_{\alpha\beta}(u), \alpha, \beta = 1, 2, 3$ are 3×3 matrices acting on the nth site of the lattice. It means that the monodromy matrix has the form

$$T(u) = \begin{pmatrix} A_1(u) & B_1(u) & B_2(u) \\ C_1(u) & A_2(u) & B_3(u) \\ C_2(u) & C_3(u) & A_3(u) \end{pmatrix} \tag{50}$$

where

$$T_{ij}(u) = \sum_{k_1,\ldots,k_{N-1}=1}^{3} L^{(N)}_{ik_1}(u) \overset{s}{\otimes} L^{(N-1)}_{k_1k_2}(u) \overset{s}{\otimes} \cdots \overset{s}{\otimes} L^{(1)}_{k_{N-1}j}(u) \tag{51}$$

$$i, j = 1, 2, 3. \tag{52}$$

The vector in the quantum space of the monodromy matrix $T(u|z)$ that is annihilated by the operators $T_{ij}(u), i > j$ ($C_i(u)$ operators, $i = 1, 2, 3$) and is also an eigenvector for the operators $T_{ii}(u)$ ($A_i(u)$ operators, $i = 1, 2, 3$) is called the highest vector of the monodromy matrix $T(u)$.

The transfer matrix $\tau(u)$ of the corresponding integrable spin model is given by the supertrace of the monodromy matrix in the space V,

$$\tau(u) = \sum_{i=1}^{3} (\varepsilon)^{p(a)} T_{ii}(u) = A_1(u) + \varepsilon A_2(u) + A_3(u). \tag{53}$$

We will define the local vacuum in a lattice of N sites as the even (bosonic) completely unoccupied state in the form

$$|0\rangle = \otimes_{a=1}^{N} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_a . \tag{54}$$

Using (51) we can compute the normalized action of the monodromy matrix entries on this state as

$$\begin{aligned} &A_i(u)|0\rangle = X_i(u)|0\rangle \qquad C_i(u)|0\rangle = 0 \qquad B_i(u)|0\rangle \neq \{0, |0\rangle\} \\ &X_i(u) = \prod_{a=1}^{N} \frac{x_i(u)}{x_2(u)} \qquad i = 1, 2, 3. \end{aligned} \tag{55}$$

Then we have the action of the transfer matrix in the local vacuum in the form

$$\tau(u)|0\rangle = \Lambda_0(u)|0\rangle$$

with the eigenvalues

$$\Lambda_0(u) = x_1^N + \varepsilon x_2^N + x_3^N. \tag{56}$$

The next step in the Bethe ansatz construction is to define the one-particle state. This state can be written in the form

$$\Psi_1(v) = B_1(v)|0\rangle. \tag{57}$$

To study the action of $\tau(u)$ in this state we need to use some commutation relations (47), because now we have the operators $A_i(u)$ acting in the form

$$\begin{aligned} \tau(u)\Psi_1(v) &= \Lambda_1(u, v)\Psi_1(v) \\ &= (A_1(u) + \varepsilon A_2(u) + A_3(u))B_1(v)|0\rangle. \end{aligned} \tag{58}$$

The necessary commutation relations between the operators $A_i(u)B_1(v)$ are

$$A_1(u)B_1(v) = z(v-u)B_1(v)A_1(u) - \frac{x_5(v-u)}{x_2(v-u)}B_1(u)A_1(v) \tag{59}$$

$$\begin{aligned} A_2(u)B_1(v) = {} & \varepsilon\frac{z(u-v)}{\omega(u-v)}B_1(v)A_2(u) - \frac{z(l-v)}{\omega(u-v)}\frac{1}{y(v-u)}B_2(v)C_1(u) \\ & - \varepsilon\frac{y_5(u-v)}{x_2(u-v)}B_1(u)A_2(v) + \frac{y_5(u-v)}{x_2(u-v)}\frac{1}{y(u-v)}B_2(u)C_1(v) \\ & + \frac{1}{y(u-v)}B_3(u)A_1(v) \end{aligned} \tag{60}$$

$$\begin{aligned} A_3(u)B_1(v) = {} & \frac{x_2(u-v)}{x_3(u-v)}B_1(v)A_3(u) - \frac{\varepsilon}{y(u-v)}B_3(u)A_2(v) \\ & + \frac{x_5(u-v)}{x_3(u-v)}B_2(v)C_3(u) - \frac{y_7(u-v)}{x_3(u-v)}B_2(u)C_3(v) \end{aligned} \tag{61}$$

where we use the following notation:

$$\begin{aligned} z(u) &= \frac{x_1(u)}{x_2(u)} \qquad & \omega(u) &= \varepsilon\frac{x_1(u)x_3(u)}{x_3(u)x_4(u) - x_6(u)y_6(u)} \\ y(u) &= \frac{x_3(u)}{y_6(u)} \qquad & y(-u) &= \varepsilon\frac{x_3(u)x_4(u) - x_6(u)y_6(u)}{x_7(u)y_6(u) - x_3(u)x_6(u)}. \end{aligned} \tag{62}$$

The action of the transfer matrix $\tau(u)$ in this state $\Psi_1(v)$ gives us

$$\begin{aligned} \tau(u)\Psi_1(v) &= (A_1(u) + \varepsilon A_2(u) + A_3(u)) \\ &= \left[z(v-u)x_1^N(u) + \varepsilon^2\frac{z(u-v)}{\omega(u-v)}x_2^N(u) + \frac{x_2(u-v)}{x_3(u-v)}x_3^N(u)\right]\Psi_1(v) \\ &\quad - \left[\frac{x_5(v-u)}{x_2(v-u)}x_1^N(u) + \varepsilon^2\frac{y_5(u-v)}{x_2(u-v)}x_2^N(u)\right]B_1(u)|0\rangle \\ &\quad + \varepsilon\left[\frac{1}{y(u-v)}x_1^N(v) - \frac{1}{y(u-v)}x_2^N(u_1)\right]B_3(u)|0\rangle. \end{aligned} \tag{63}$$

Then the eigenvalue has the form

$$\Lambda_1(u,v) = \left[z(v-u)x_1^N(u) + \varepsilon^2\frac{z(u-v)}{\omega(u-v)}x_2^N(u) + \frac{x_2(u-v)}{x_3(u-v)}x_3^N(u)\right] \tag{64}$$

and the Bethe equations have the form

$$\left(\frac{x_1(v)}{x_2(v)}\right)^N = \varepsilon^2 = 1 \tag{65}$$

where we use the identity

$$\frac{x_5(u)}{x_2(u)} = -\frac{y_5(-u)}{x_2(-u)}.$$

The next step is to define the two-particle state and for this we can use the fundamental relation (47) where we will have

$$B_1(u)B_1(v) = \omega(v-u)\left[B_1(v)B_1(u) - \frac{1}{y(v-u)}B_2(v)A_1(u)\right] + \frac{1}{y(u-v)}B_2(u)A_1(v). \tag{66}$$

It is easy to observe that (66) satisfies the condition

$$\Psi_2(v_2, v_1) = \omega(v_1 - v_2)\Psi_2(v_1, v_2). \tag{67}$$

This gives us the definition for the two-particle state $\Psi_2(v_2, v_1)$ as

$$\Psi_2(v_1, v_2) = B_1(v_1)B_1(v_2)|0\rangle - \frac{1}{y(v_1 - v_2)}B_2(v_1)A_1(v_2)|0\rangle. \tag{68}$$

To study the action of the transfer matrix $\tau(u)$ in this state we will need some more commuting relations:

$$A_1(\lambda)B_2(\mu) = \frac{x_1(\mu-\lambda)}{x_3(\mu-\lambda)}B_2(\mu)A_1(\lambda) - \frac{x_7(\mu-\lambda)}{x_3(\mu-\lambda)}B_2(\lambda)A_1(\mu) - \varepsilon\frac{x_6(\mu-\lambda)}{x_3(\mu-\lambda)}B_1(\lambda)B_1(\mu)$$

$$A_2(\lambda)B_2(\mu) = z(\lambda-\mu)z(\mu-\lambda)B_2(\mu)A_2(\lambda) + \frac{y_5(\lambda-\mu)}{x_2(\lambda-\mu)}\left[B_1(\lambda)B_3(\mu) - \varepsilon B_3(\lambda)B_1(\mu) + \frac{y_5(\lambda-\mu)}{x_2(\lambda-\mu)}B_2(\lambda)A_2(\mu)\right]$$

$$A_3(\lambda)B_2(\mu) = \frac{x_1(\lambda-\mu)}{x_3(\lambda-\mu)}B_2(\mu)A_3(\lambda) - \frac{y_7(\lambda-\mu)}{x_3(\lambda-\mu)}B_2(\lambda)A_3(\mu) - \frac{\varepsilon}{y(\lambda-\mu)}B_3(\lambda)B_3(\mu)$$

$$C_1(\lambda)B_1(\mu) = \varepsilon B_1(\mu)C_1(\lambda) + \frac{y_5(\lambda-\mu)}{x_2(\lambda-\mu)}[A_1(\mu)A_2(\lambda) - A_1(\lambda)A_2(\mu)]$$

$$C_3(\lambda)B_1(\mu) = \varepsilon\frac{x_4(\lambda-\mu)}{x_3(\lambda-\mu)}B_1(\mu)C_3(\lambda) - \frac{x_7(\lambda-\mu)}{x_3(\lambda-\mu)}B_1(\lambda)C_3(\mu) + \frac{1}{y(\lambda-\mu)}[A_1(\mu)A_3(\lambda) - A_2(\lambda)A_2(\mu)] + \frac{x_6(\lambda-\mu)}{x_3(\lambda-\mu)}B_2(\mu)C_2(\lambda)B_1(\lambda)$$

$$B_1(\lambda)B_2(\mu) = \frac{1}{z(\lambda-\mu)}B_2(\mu)B_1(\lambda) + \frac{y_5(\lambda-\mu)}{x_1(\lambda-\mu)}B_1(\mu)B_2(\lambda)$$

$$B_1(\lambda)B_3(\mu) = \varepsilon B_3(\mu)B_1(\lambda) - \frac{y_5(\lambda-\mu)}{x_2(\lambda-\mu)}B_2(\mu)A_2(\lambda) + \frac{x_5(\lambda-\mu)}{x_2(\lambda-\mu)}B_2(\lambda)A_2(\mu)$$

$$B_2(\lambda)B_1(\mu) = \frac{1}{z(\lambda-\mu)}B_1(\mu)B_2(\lambda) + \frac{x_5(\lambda-\mu)}{x_1(\lambda-\mu)}B_2(\mu)B_1(\lambda).$$

In this state we will have the eigenstate as

$$\Lambda_2(u, v_1, v_2) = z(v_{10})z(v_{20})x_1^N(u) + \varepsilon^2\frac{z(v_{01})}{\omega(v_{01})}\frac{z(v_{02})}{\omega(v_{02})}x_2^N(u) + \frac{x_2(v_{01})}{x_3(v_{01})}\frac{x_2(v_{02})}{x_3(v_{02})}x_3^N(u) \tag{69}$$

and the Bethe equations have the form

$$\left(\frac{x_1(v_a)}{x_2(v_a)}\right)^N = \varepsilon^2\frac{z(v_{ab})}{z(v_{ba})}\omega(v_{ba}) \qquad a \neq b = 1, 2 \tag{70}$$

where $v_{ab} = v_a - v_b$, $a \neq b = 0, 1, 2$, and $u_0 = u$.

Proceeding with the method we will find that for the n-particle state the eigenvalue has the form [24]

$$\Lambda_M = x_1(u)^N\prod_{a=1}^{M} z(u_a - u) + \varepsilon^{M+1}x_2(u)^N\prod_{a=1}^{M}\frac{z(u-u_a)}{\omega(u-u_a)} + x_3(u)^N\prod_{a=1}^{M}\frac{x_2(u-u_a)}{x_3(u-u_a)} \tag{71}$$

and the Bethe equations

$$\left(\frac{x_1(u_a)}{x_2(u_a)}\right)^N = \varepsilon^{M+1}\prod_{b\neq a=1}^{M}\frac{z(u_a-u_b)}{z(u_b-u_a)}\omega(u_b-u_a) \qquad a = 1, 2, \ldots, M. \tag{72}$$

Many of the relations found in this section will be used in the next section with reflection condition. It is important to observe that the method for the resolution of the eigenvalue problem is the same, but we will have some more complex relations because in the case with reflection condition we will lose the translational invariance.

4. Algebraic Bethe ansatz with reflection condition

To begin to study the case with reflection condition we need the so-called reflection equation (5)

$$\mathcal{R}_{12}(u-v)K_1(u)\mathcal{R}_{21}(u+v)K_2(v) = K_2(v)\mathcal{R}_{12}(u+v)K_1(u)\mathcal{R}_{21}(u-v). \tag{73}$$

Assuming that $K(u)$ is a solution of the reflection equation (73) we can define the double-row monodromy matrix as

$$U(u) = T(u)K^-(u)T^{-1}(-u). \tag{74}$$

For the algebraic BA with reflection matrices, the fundamental relation has the form

$$R_{12}(u-v)U_1(u)R_{21}(u+v)U_2(v) = U_2(v)R_{12}(u+v)U_1(u)R_{21}(u-v). \tag{75}$$

From (1) and (75) we can define the transfer matrix as

$$\tau(u) = \operatorname{tr} K^+U(u). \tag{76}$$

In section 2 we write the diagonal solutions of (73) for the 19-vertex models and the form of the $K^-(u)$ and $K^+(u)$ was also presented.

From (2) we can define the $T^{-1}(-u)$ monodromy matrix as

$$T^{-1}(-u) = \mathcal{L}_{N,\alpha}(u)\mathcal{L}_{N-1,\alpha}(u)\dots\mathcal{L}_{1,\alpha}(u)$$

that has a similar form to (50)

$$T^{-1}(-u) = \begin{pmatrix} \overline{A}_1(u) & \overline{B}_1(u) & \overline{B}_2(u) \\ \overline{C}_1(u) & \overline{A}_2(u) & \overline{B}_3(u) \\ \overline{C}_2(u) & \overline{C}_3(u) & \overline{A}_3(u) \end{pmatrix}.$$

It is very easy to show that each entry of $T^{-1}(-u)$ acts in the local vacuum in a form similar to that of the matrix $T(u)$:

$$\overline{A}_1(u)|0\rangle = x_1^N(u)|0\rangle \qquad \overline{A}_2(u)|0\rangle = x_2^N(u)|0\rangle \qquad \overline{A}_3(u)|0\rangle = x_3^N(u)|0\rangle$$
$$\overline{C}_i(u)|0\rangle = 0$$
$$\overline{B}_i(u)|0\rangle \neq 0 \qquad i = 1, 2, 3.$$

We can now write (74) as

$$U(u) = T(u)K^-(u)T^{-1}(-u) = \begin{pmatrix} \mathcal{A}_1(u) & \mathcal{B}_1(u) & \mathcal{B}_2(u) \\ \mathcal{C}_1(u) & \mathcal{A}_2(u) & \mathcal{B}_3(u) \\ \mathcal{C}_2(u) & \mathcal{C}_3(u) & \mathcal{A}_3(u) \end{pmatrix} \tag{77}$$

where each of the entries of (77) has the form

$$\mathcal{A}_1(u) = k_{11}^- A_1(u)\overline{A}_1(u) + k_{22}^- B_1(u)\overline{C}_1(u) + k_{33}^- B_2(u)\overline{C}_2(u)$$
$$\mathcal{A}_2(u) = k_{11}^- C_1(u)\overline{B}_1(u) + k_{22}^- A_2(u)\overline{A}_2(u) + k_{33}^- B_3(u)\overline{C}_3(u)$$
$$\mathcal{A}_3(u) = k_{11}^- C_2(u)\overline{B}_2(u) + k_{22}^- C_3(u)\overline{B}_3(u) + k_{33}^- A_3(u)\overline{A}_3(u)$$

$$\mathcal{B}_1(u) = k_{11}^- A_1(u)\overline{B}_1(u) + k_{22}^- B_1(u)\overline{A}_2(u) + k_{33}^- B_2(u)\overline{C}_3(u)$$
$$\mathcal{B}_2(u) = k_{11}^- A_1(u)\overline{B}_2(u) + k_{22}^- B_1(u)\overline{B}_3(u) + k_{33}^- B_2(u)\overline{A}_3(u)$$
$$\mathcal{B}_3(u) = k_{11}^- C_1(u)\overline{B}_2(u) + k_{22}^- A_2(u)\overline{B}_3(u) + k_{33}^- B_3(u)\overline{A}_3(u)$$

$$\mathcal{C}_1(u) = k_{11}^- C_1(u)\overline{A}_1(u) + k_{22}^- A_2(u)\overline{C}_1(u) + k_{33}^- B_3(u)\overline{C}_2(u)$$
$$\mathcal{C}_2(u) = k_{11}^- C_2(u)\overline{A}_1(u) + k_{22}^- C_3(u)\overline{C}_1(u) + k_{33}^- A_3(u)\overline{C}_2(u)$$
$$\mathcal{C}_3(u) = k_{11}^- C_2(u)\overline{B}_1(u) + k_{22}^- C_3(u)\overline{A}_2(u) + k_{33}^- A_3(u)\overline{C}_3(u).$$

To study how the entries of the double-row monodromy matrix (74) act in the local vacuum we need to use the graded YB equation (43) where we can rewrite the equation in the form

$$T_2^{-1}(v)R_{12}(u-v)T_1(u) = T_1(u)R_{12}(u-v)T_2^{-1}(v) \tag{78}$$

where $T_1 = I \otimes T$, $T_2 = T \otimes I$ and I represents the 3×3 identity matrix. If in the relations generated in (78) we put $v = -u$, we will get a set of commuting relations between the entries of the $T(u)$ and $T^{-1}(u)$ monodromy matrices. For example,

$$\overline{C}_1(u)B_1(u) = -\frac{x_5(2u)}{x_1(2u)}\overline{A}_2(u)A_2(u) + \frac{x_5(2u)}{x_1(2u)}\overline{A}_1(u)A_1(u).$$

With these relations we can reorder the operators (77) and it is now possible to show that they act in the local vacuum $|0\rangle$ in the form

$$\mathcal{A}_1(u)|0\rangle = x_1^{2N}(u)|0\rangle \qquad \mathcal{A}_2(u)|0\rangle = x_2^{2N}(u)|0\rangle \qquad \mathcal{A}_3(u)|0\rangle = x_3^{2N}(u)|0\rangle$$
$$\mathcal{C}_i(u)|0\rangle = 0$$
$$\mathcal{B}_i(u)|0\rangle \neq 0 \qquad i = 1, 2, 3.$$

Other useful information that we can obtain from (78) is the possibility of rewriting the diagonal operators $\mathcal{A}_i(u)$ in a form that will let us rewrite the commuting relation only with one wanted term (i.e. only with terms of type $B_1(v)\mathcal{A}_i(u)$) and this fact will simplify the execution of the algebraic Bethe ansatz.

Then, from now on, we will define the diagonal operators as

$$A_1(u) = A_1(u)$$
$$\widetilde{A}_2(u) = A_2(u) - \frac{y_5(2u)}{x_1(2u)}A_1(u)$$
$$\widetilde{A}_3(u) = A_3(u) + \frac{y_5(2u)x_1(2u) - y_7(2u)x_5(2u)}{y_5(2u)x_5(2u) - \varepsilon x_4(2u)x_1(2u)}\widetilde{A}_2(u) - \frac{y_7(2u)}{x_1(2u)}A_1(u)$$

or in a more compact form

$$\begin{aligned} A_1(u) &= \mathcal{A}_1(u) \\ \widetilde{A}_2(u) &= \mathcal{A}_2(u) + \Gamma_1 A_1(u) \\ \widetilde{A}_3(u) &= \mathcal{A}_3(u) + \Gamma_2\widetilde{A}_2(u) + \Gamma_3 A_1(u) \end{aligned} \tag{79}$$

where

$$\Gamma_1 = \frac{y_5(2u)}{x_1(2u)} \qquad \Gamma_3 = -\frac{y_7(2u)}{x_1(2u)}$$
$$\Gamma_2 = \frac{y_5(2u)x_1(2u) - y_7(2u)x_5(2u)}{y_5(2u)x_5(2u) - \varepsilon x_4(2u)x_1(2u)}.$$

We will write the action of the diagonal operators (79) in the local vacuum as

$$\begin{aligned} &A_1(u)|0\rangle = \Delta_1(u)|0\rangle \qquad \widetilde{A}_2(u)|0\rangle = \Delta_2(u)|0\rangle \\ &\widetilde{A}_3(u)|0\rangle = \Delta_3(u)|0\rangle. \end{aligned} \tag{80}$$

In all the following relations we will use the diagonal operators as written in (79).

As in the case with a periodic boundary condition, we can define the one-particle state as

$$\Psi_1(v) = \mathcal{B}_1(v)|0\rangle.$$

Now the transfer matrix has the form (9)

$$\tau(u) = k_{11}^{+}(u)\mathcal{A}_1(u) + \varepsilon k_{22}^{+}(u)\mathcal{A}_2(u) + k_{33}^{+}(u)\mathcal{A}_3(u).$$

But from (79) we have

$$\begin{aligned}\tau(u) &= k_{11}^{+}(u)\mathcal{A}_1(u) + \varepsilon k_{22}^{+}(u)(\widetilde{A}_2(u) - \Gamma_1 A_1(u)) + k_{33}^{+}(u)(\widetilde{A}_3(u) - \Gamma_2\widetilde{A}_2(u) - \Gamma_3 A_1(u)) \\ &= \left(k_{11}^{+}(u) + \varepsilon k_{22}^{+}(u)\Gamma_1 - k_{33}^{+}\Gamma_3\right) A_1(u) + \left(\varepsilon k_{22}^{+}(u) - k_{33}^{+}(u) - \Gamma_2\right)\widetilde{A}_2(u) + k_{33}^{+}(u)\widetilde{A}_3(u)\end{aligned}$$

or in a more compact notation [27]

$$\tau(u) = w_1^{+} A_1(u) + w_2^{+}\widetilde{A}_2(u) + w_3^{+}\widetilde{A}_3(u). \tag{81}$$

To study the action of the transfer matrix (81) in the one-particle state $\Psi_1(v)$ we will need the following relation (note that in the case of a one-particle state we will not need the terms $B_2(u)C_i(v)$, but they will be necessary in the study of states with more particles).

It is important to note that it is straightforward to obtain the commutation relation, because it is necessary to use in many cases two or more relations from (75) and also to redefine the diagonal operators from the relations obtained before. After some tedious algebraic manipulation this construction gives very complicate coefficients for the commutation relations

$$\begin{aligned}A_1(u)B_1(v) = {}& \frac{h(-)}{z(+)}B_1(v)A_1(u) + \left(\frac{x_5(-)}{z(+)x_2(-)} - \beta_1\frac{x_5(+)}{x_1(+)}\right)B_1(u)A_1(v) \\ & - \frac{x_7(+)}{x_1(+)}B_2(u)C_3(v) + \varepsilon\frac{x_6(+)h(-)}{x_1(+)}B_2(v)C_1(u) \\ & - \frac{x_5(+)}{x_1(+)}B_1(u)\widetilde{A}_2(v) + \frac{x_6(+)x_5(-)}{x_1(+)x_2(-)}B_2(u)C_1(v)\end{aligned}$$

$$\begin{aligned}\widetilde{A}_2(u)B_1(v) = {}& \frac{1}{z(2v)}\left(\frac{x_5(+)}{x_1(+)} - \Gamma_1\frac{x_1(+)}{x_5(+)}\frac{y(-)}{y(+)}\right)B_1(u)A_1(v) \\ & + \frac{y_5(-)}{x_2(-)}\left(\Gamma_1\frac{y_5(+)x_7(+)}{x_2(+)x_1(+)} - \frac{x_5(+)}{x_2(+)}\right)B_2(u)C_3(v) \\ & - \varepsilon\frac{x_6(+)x_5(-)}{x_1(+)x_2(-)}\left(\Gamma_1 - \frac{y_5(-)}{x_2(-)}\right)B_2(u)c_1(v) \\ & + \frac{x_5(-)}{x_2(-)}\left(\Gamma_1\frac{x_3(-)}{x_2(-)} - 1\right)B_2(v)C_1(u) \\ & + \left(\Gamma_1\frac{x_5(+)}{x_1(+)} + \frac{1}{\omega(+)}\frac{y_5(-)}{x_2(-)}\right)B_1(u)\widetilde{A}_2(v) \\ & + \frac{1}{z(2v)}\frac{y_6(-)}{x_2(-)}B_3(u)A_1(v) + \frac{t^2(+)}{w(-)}B_1(v)\widetilde{A}_2(u) \\ & + \frac{1}{w(-)}\left(\frac{y_5(+)x_7(+)}{x_2(+)x_1(+)} - \frac{x_5(+)}{x_2(+)}\right)B_2(v)C_3(u) - \varepsilon\frac{x_6(+)}{x_2(+)}B_3(u)\widetilde{A}_2(v).\end{aligned}$$

And the last diagonal operator $\widetilde{A}_3(u)$ has a more complicated coefficient:

$$\begin{aligned}\widetilde{A}_3(u)B_1(v) = {}& f_1 B_1(v)\widetilde{A}_3(u) + f_2 B_3(u)\widetilde{A}_2(v) + f_3 B_3(u)A_1(v) + f_4 B_1(u)\widetilde{A}_2(v) \\ & + f_5 B_1(u)A_1(v) + f_6 B_2(u)C_3(v) + f_7 B_2(u)C_1(v)\end{aligned}$$

where

$$f_1 = \frac{x_2(-)}{x_3(-)}\left(\frac{x_2(+)}{x_3(+)} - \varepsilon\frac{y_6(+)x_6(+)}{x_3(+)x_2(+)}\right)$$

$$f_2 = \frac{1}{y(-)}\left(\frac{x_2(+)}{x_3(+)} - \varepsilon\frac{y_6(+)x_6(+)}{x_3(+)x_2(+)}\right) + \varepsilon\Gamma_2\frac{x_6(+)}{x_2(+)}$$

$$f_3 = \frac{x_6(+)}{x_2(+)}\left(\frac{x_2^2(-)}{x_3^2(-)} - \frac{y_6(-)x_6(-)}{x_3^2(-)}\right) + \Gamma_2\left(\frac{x_6(-)x_3(+)}{x_3(-)x_2(+)} + \Gamma_1(2v)\frac{x_6(+)}{x_2(+)}\right)$$

$$f_4 = t(+)\frac{y_5(-)}{x_2(-)}\left(\Gamma_2 - \varepsilon\frac{y_6(-)y_6(+)}{x_3(-)x_3(+)}\right) - \frac{y_7(-)}{x_3(-)}l(+)$$

$$\begin{aligned} f_5 = \Gamma_2\Bigg[&\frac{y_5(+)}{x_1(+)}\left(\frac{\overline{h}(-)}{w(-)} - \frac{y_5(-)x_5(-)}{x_2^2(-)}\right) - \frac{y_5(-)}{x_2(-)}t(+)t(-)\beta_1\Bigg] \\ &+ \frac{y_7(+)x_2(+)}{x_3(+)x_1(+)}\left(\frac{y_5(-)h(-)}{x_3(-)} - \frac{y_7(-)x_5(-)}{x_3(-)x_2(-)}\right) \\ &+ \frac{y_6(+)y_5(+)}{x_3(+)x_1(+)}\left[-\varepsilon\frac{y_6(-)}{x_3(-)}\left(\frac{h(-)}{w(-)} - \frac{y_5(-)x_5(-)}{x_2(-)x_1(-)} - \frac{x_2(-)}{x_3(-)}\right)\right] \end{aligned}$$

$$f_6 = \Gamma_2\left[\frac{y_5(-)y_5(+)}{x_2(-)x_2(+)}\left(\frac{x_7(+)}{x_1(+)} + \Gamma_3\right)\right] - l(+)\left(\frac{y_7(-)}{x_3(-)} - \varepsilon\frac{y_6(-)y_5(-)y_6(+)}{x_3(-)x_2(-)x_3(+)}\right)$$

$$f_7 = \Gamma_2\frac{x_6(+)y_5(+)}{x_1(+)x_2(+)}\left(\frac{\overline{t}(-)}{w(-)} + \varepsilon\frac{y_5(-)x_5(-)}{x_2^2(-)}\right) - \Gamma_3\frac{x_5(-)x_6(+)}{x_2(-)x_2(+)}$$

and we introduce the new functions

$$h(u) = \frac{x_2^2(u) - x_5(u)y_5(u)}{x_1(u)x_2(u)} \qquad l(u) = \frac{y_7(u)x_5(u) - y_5(u)x_1(u)}{x_1(u)x_3(u)}$$

$$t(u) = \frac{y_5(u)x_5(u) - \varepsilon x_1(u)x_4(u)}{x_1(u)x_2(u)}.$$

For the one-particle state we have the eigenvalue

$$\Lambda_1(u,v) = w_1^+\frac{h(-)}{z(+)}\Delta_1(u) + w_2^+\frac{t^2(+)}{w(-)}\Delta_2(u) + w_3^+\frac{x_2(-)}{x_3(-)}\left(\frac{x_2(+)}{x_3(+)} - \varepsilon\frac{y_6(+)x_6(+)}{x_3(+)x_2(+)}\right)\Delta_3(u)$$

and the Bethe equations in the form

$$\left(\frac{\Delta_1(v)}{\Delta_2(v)}\right)^{2N} = 1.$$

To study the two-particle state $\Psi_2(v_1, v_2)$ we need first to define this state, and for this we will use the fundamental relation (75) where we can find the form for $\Psi_2(v_1, v_2)$ as

$$\Psi_2(v_1,v_2) = B_1(v_1)B_1(v_2) + \varepsilon\frac{x_6(+)}{x_2(+)}B_2(v_1)A_2(v_2) - \varepsilon\frac{x_6(-)x_3(+)}{x_3(-)x_2(+)}B_2(v_1)A_1(v_2)$$

where we will use the following notation from here $x_i(-) = x_i(u-v)$, $\overline{x}_i(-) = x_i(v-u)$ and $x_i(+) = x_i(u+v)$.

As in the case of the periodic boundary condition, this state satisfies the important relation

$$\Psi_2(v_2,v_1) = w(v_1 - v_2)\Psi_2(v_1,v_2)$$

where $w(v_1 - v_2)$ is the same function defined in the periodic boundary condition (62).

To study the action of the transfer matrix in the two-particle state it is useful to have some new relations:

$$C_1(u)B_1(v) = \frac{x_5(+)}{x_1(+)}\left(1-\Gamma_1\frac{y_5(-)}{x_2(-)}\right)A_1(u)A_1(v)+\frac{\overline{x}_5(-)}{\overline{x}_2(-)z(+)}\widetilde{A}_2(v)A_1(u)$$
$$+\Gamma_1(2v)\left(\frac{\overline{x}_5(-)x_2(+)}{\overline{x}_2(-)x_1(+)}\right)A_1(v)A_1(u)+\varepsilon\frac{x_4(+)}{x_1(+)}B_1(v)C_1(u)$$
$$+\varepsilon\frac{x_5(-)x_6(+)}{x_2(-)x_1(+)}B_3(v)C_1(u)+\frac{x_5(+)}{x_1(+)}B_2(v)C_2(u)$$
$$-\varepsilon\frac{\overline{y}_5(-)x_6(+)}{\overline{x}_2(-)x_1(+)}B_3(u)C_1(v)-\frac{x_7(+)}{x_1(+)}B_3(u)C_3(v)$$
$$-\frac{x_5(+)}{x_1(+)}\widetilde{A}_2(u)\widetilde{A}_2(v)-\Gamma_1(2u)\frac{x_5(+)}{x_1(+)}A_1(u)\widetilde{A}_2(v)$$

$$C_3(u)B_1(v) = \varepsilon f(+)\left(\frac{x_4(-)}{x_6(-)}+\frac{y_6(-)}{x_3(-)}\right)B_3(v)C_3(u)+\varepsilon\frac{x_4(-)}{x_6(-)}f(+)B_3(u)C_3(v)$$
$$-\frac{x_4(-)y_5(+)x_4(+)}{x_6(-)x_2(+)x_1(+)}B_1(u)C_1(v)+t(+)\left(\frac{x_4(-)}{x_6(-)}+\varepsilon\frac{y_6(-)}{x_3(-)}\right)B_1(u)C_1(v)$$
$$+\frac{y_6(-)}{x_3(-)}t(+)B_1(v)C_1(u)-\frac{y_6(-)}{x_6(-)}B_1(u)C_3(v)+t(+)\frac{x_4(-)}{x_6(-)}\widetilde{A}_2(u)\widetilde{A}_2(v)$$
$$+\varepsilon\frac{x_4(-)y_5(+)x_5(+)}{x_6(-)x_2(+)x_1(+)}B_2(v)C_2(u)+t(+)\left(\frac{x_4(-)}{x_6(-)}-\frac{y_6(-)}{x_3(-)}\right)B_2(v)C_2(u)$$
$$-\varepsilon\frac{x_6(+)}{x_2(+)}\widetilde{A}_3(u)\widetilde{A}_2(v)+\varepsilon\left(\frac{x_6(-)x_3(+)}{x_3(-)x_2(+)}-\Gamma(2v)\frac{x_6(+)}{x_2(+)}\right)\widetilde{A}_3(u)A_1(v)$$

$$B_1(u)B_2(v) = \left(\frac{x_5(+)}{x_3(+)}-\frac{y_5(+)y_7(+)}{x_3(+)x_1(+)}\right)B_2(v)B_1(u)$$
$$-\varepsilon\frac{x_6(+)}{x_2(+)}B_2(u)B_3(v)+\frac{x_5(-)x_3(+)}{x_2(-)x_2(+)}B_2(u)B_1(v).$$

For this state we have the eigenvalues given as

$$\Lambda_2(u,v_1,v_2) = w_1^+\left[\frac{x_2(u+v_1)}{x_1(u+v_1)}\left(\frac{x_5(u-v_1)y_5(u-v_1)}{x_1(u-v_1)x_2(u-v_1)}+\frac{x_2(u-v_1)}{x_1(u-v_1)}\right)\right.$$
$$\left.\times\ \frac{x_2(u+v2)}{x_1(u+v2)}\left(\frac{x_5(u-v_2)y_5(u-v_2)}{x_1(u-v_2)x_2(u-v_2)}+\frac{x_2(u-v_2)}{x_1(u-v_2)}\right)\right]\Delta_1(u)$$
$$+\varepsilon w_2^+\left[\frac{z(u-v_1)}{\omega(u-v_1)}\left(\frac{x_4(u+v_1)}{x_2(u+v_1)}-\varepsilon\frac{y_5(u+v_1)x_5(u+v_1)}{x_2(u+v_1)x_1(u+v_1)}\right)\right.$$
$$\left.\times\ \frac{z(u-v_2)}{\omega(u-v_2)}\left(\frac{x_4(u+v_2)}{x_2(u+v_2)}-\varepsilon\frac{y_5(u+v_2)x_5(u+v_2)}{x_2(u+v2)x_1(u+v_2)}\right)\right]\Delta_2(u)$$
$$+w_3^+\left[\frac{x_2(u-v_1)}{x_3(u-v_1)}\left(\frac{x_2(u+v_1)}{x_3(u+v_1)}-\varepsilon\frac{y_6(u+v_1)x_6(u+v_1)}{x_3(u+v_1)x_2(u+v_1)}\right)\right.$$
$$\left.\times\ \frac{x_2(u-v_2)}{x_3(u-v_2)}\left(\frac{x_2(u+v_2)}{x_3(u+v_2)}-\varepsilon\frac{y_6(u+v_2)x_6(u+v_2)}{x_3(u+v_2)x_2(u+v_2)}\right)\right]\Delta_3(u)$$

and the Bethe equations

$$\left(\frac{\Delta_1(v_i)}{\Delta_2(v_i)}\right)^{2N} = -\varepsilon\frac{w_1^+}{w_2^+}\left\{\left(\frac{x_1(-)}{x_3(-)x_2^2(+)}\right)\right.$$
$$\left.\times\left(\frac{\left(x_4(-)x_3(-)-x_6(-v_j)y_6(-)\right)}{(x_2(-)x_2(-)-x_5(-)y_5(-))}(\varepsilon y_5(+)x_5(+)-x_4(+)x_1(+))\right)\right\}$$
$$x_k(-)=x_{ki}(v_i-v_j) \qquad x_k(+)=x_i(v_i+v_j) \qquad i\neq j=1,2.$$

Proceeding with the constructive method we will find that for a general n-particle state $\Psi_n(v_1, v_2, \ldots, v_n)$ we have for the eigenvalues:

$$\Lambda = w_1^+\prod_{i=1}^n\frac{x_2(+)}{x_1(+)}\left(\frac{x_5(-)y_5(-)}{x_1(-)x_2(-)}+\frac{x_2(-)}{x_1(-)}\right)\Delta_1(u)$$
$$+\varepsilon^{n+1}w_2^+\prod_{i=1}^n\frac{z(-)}{\omega(-)}\left(\frac{x_4(+)}{x_2(+)}-\varepsilon\frac{y_5(+)x_5(+)}{x_2(+)x_1(+)}\right)\Delta_2(u)$$
$$+w_3^+\prod_{i=1}^n\frac{x_2(-)}{x_3(-)}\left(\frac{x_2(+)}{x_3(+)}-\varepsilon\frac{y_6(+)x_6(+)}{x_3(+)x_2(+)}\right)\Delta_3(u). \qquad (82)$$

And the Bethe equations can be written as

$$\left(\frac{\Delta_1(v_i)}{\Delta_2(v_i)}\right)^{2N} = -\varepsilon^{n+1}\prod_{\substack{i=1\\ j\neq i}}^n\frac{w_1^+}{w_2^+}\left\{\left(\frac{x_1(-)}{x_3(-)}\right)\right.$$
$$\left.\times\left(\frac{(x_4(-)x_3(-)-x_6(-)y_6(-))}{(x_2(-)x_2(-)-x_5(-)y_5(-))}\left(\frac{\varepsilon y_5(+)x_5(+)-x_4(+)x_1(+)}{x_2(+)x_2(+)}\right)\right)\right\} \qquad (83)$$
$$i=1,\ldots,n.$$

It is important to note that the mapping between the 19-vertex models presented in section 2 can be very useful to obtain one solution from the others.

5. Spin one chains derived from 19-vertex models

In [32] quantum spin chains of spin one derived from the 19-vertex models were classified. One of these models is the ZF model (solution 10 for the trigonometric version and solution 7 for the rational version), another is the $su(3)$-invariant model [33] that was obtained in [34] (solution 4). Solution 7 was obtained in [35, 36]. Solution 5 appears in [37] and solution 6 in [38]. Solutions 2, 3, 8 and 9 were new. From these new solutions we have solvable *t–J* models (solutions 1–4) as special cases. The following tables give the classified solutions (in the tables we have that A, O, C are arbitrary constants and $v \equiv e^u$, $i_k = \pm 1$, $a^2+a-1=0$, $b^2-(a-1)/b-(a-1)^2=0$):

Vertex weight	#1	#2	#3
$x_1(u)$	e^{Au}	$\cosh(u)+\cosh(n)\sinh(u)$	$\cosh(u)+\cosh(n)\sinh(u)$
$x_2(u)$	0	$\sinh(u)\sinh(n)$	$\sinh(u)\sinh(n)$
$x_3(u)$	0	0	0
$x_4(u)$	e^{Ou}	$\cosh(u)+\cosh(n)\sinh(u)$	$\cosh(u)-\cosh(n)\sinh(u)$
$x_5(u)$	1	1	1
$x_6(u)$	0	0	0
$x_7(u)$	e^{Cu}	$\cosh(u)+\cosh(n)\sinh(u)$	$\cosh(u)+\cosh(n)\sinh(u)$

and

#4	#5	#6	#7	#8	#9
$1+i_1u$	1	1	$1+\frac{3}{2}u+\frac{1}{2}u^2$	$\frac{1}{3}v(4-v^2)$	v
i_4u	0	0	$i_2\frac{1}{2}(-u+u^2)$	0	0
i_2u	$\frac{(a-1)(1-\mathrm{e}^u)}{1+(a+b-1)\mathrm{e}^u}$	$\frac{-\mathrm{e}^u+\mathrm{e}^{2u}}{2-\mathrm{e}^u}$	$\frac{1}{2}(u+u^2)$	$-\frac{2}{3}v(1-v^4)$	$-\frac{1-v^4}{v^3\left(1-\frac{3+\sqrt{5}}{2}v^4\right)}$
$1+i_3u$	$1+b$	$b+\mathrm{e}^u$	$1-\frac{1}{2}u+\frac{1}{2}u^2$	3	$\frac{1+\sqrt{5}}{2}\frac{\left(\frac{3-\sqrt{5}}{2}+\frac{-1+\sqrt{5}}{2}v^4\right)}{v\left(-1+\frac{3+\sqrt{5}}{2}v^4\right)}$
1	1	1	$1-u$	$\frac{1}{3}(4-v^4)$	1
1	$\pm b$	$\pm b$	i_1u	$\pm\frac{\sqrt{2}}{3}v^2(1-v^4)$	$\pm\left(\frac{1+\sqrt{5}}{2}\right)^{1/2}v.b$
1	$1+b$	$b+\mathrm{e}^u$	1	$\frac{1}{3}v(2+v^4)$	$b+\frac{1}{v^3}$

The weights of solution 10 are the trigonometric Fadeev–Zamolodchikov model as described in session 2.

From these models one special model, the $su(3)$-spin model, was studied in detail in [39]. In [40] the model was studied using the so-called thermodynamic Bethe ansatz, the model can be obtained as a limit, $n \to 0$, of the A_2^2 model (Izergin–Korepin model). The model was also studied with the nested Bethe ansatz formalism in [41] and with the functional Bethe ansatz in [42] (a extension of this work using the fusion techniques for all A_{n-1}^1 models was studied in [43]).

In special interest we have studied this model with boundary conditions. In [43], Doikou solved the model with the fusion technique and with the functional Bethe ansatz approach.

The $su(3)$-invariant spin model is defined by the $R(u)$-matrix

$$\begin{aligned}
&R(u)_{jj,jj} = u+n \\
&R(u)_{jk,jk} = u \qquad j \neq k \\
&R(u)_{jk,kj} = n \qquad j \neq k \\
&1 \leqslant j,k \leqslant 3.
\end{aligned}$$

This model is a 15-vertex model, and as noted before, it can be obtained as a special limit of the Izergin–Korepin model and corresponds to the solution 4 of the above table. In [43] the Bethe equations obtained have the form

$$\left(\frac{v_i+n}{v_i-n}\right)^{2N}\left(\frac{2v_i-n}{2v_i+n}\right) = \prod_{i\neq j=1}^{n}\left(\frac{v_j-v_i-n}{v_j+v_i+n}\right)\left(\frac{v_j-v_i+2n}{v_j-v_i-2n}\right)\left(\frac{v_j+v_i-n}{v_j+v_i+n}\right)\left(\frac{v_j+v_i+2n}{v_j+v_i-2n}\right).$$

The main problem in solving this model (with this $R(u)$-matrix) with the approach used in this paper resides in the fact that the simplicity of the $R(u)$-matrix causes the loss of many relations obtained in the reflection equations (75), and in special case we lost the commutation relation between $B_1(u)B_1(v)$ and this relation is fundamental to the method used here (from this relation we derived the exact form of the two-state function $\Psi_2(u_1,u_2)$). Many of the commutation relations were obtained as a combination of two or more relations from (75). Then with the loss of many relations it is not possible to obtain the correct expression for the commutation relation.

This model can be solved by using instead the so-called *nested* Bethe ansatz [39]. In this approach we define the reference state for some of the elementary operators, and in a second stage define another reference state for the other elementary operators. With a combination of these reference states we can find a general state for the transfer matrix. Some problems in solving this model occur with all *t–J* models too (see, for example, [45] for the solution of the 15-vertex $sl(2|1)$ model).

We can instead study the rational limit of the IK model using the $R(u)$-matrix presented in this paper or the gauge version presented in [27]. In this case, the entries of the $R(u)$-matrix have the form

$$x_1(u) = u + 5n \qquad x_2(u) = -u - 3n \qquad x_3(u) = u + n \qquad x_4(u) = -u - 3n$$
$$x_5(u) = y_5(u) = 3n \qquad x_6(u) = y_6(u) = 2nI \qquad x_7(u) = y_7(u) = 4n.$$

We can write the reflection matrix as

$$K^-(u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3n-2u}{3n+2u} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad M = \begin{pmatrix} -\mathrm{e}^{\mathrm{i}4n} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\mathrm{e}^{-\mathrm{i}4n} \end{pmatrix}.$$

Using (84) we have that the Bethe equations have the form

$$\left(\frac{v_i + n}{v_i - n}\right)^{2N} \left(\frac{2v_i + n}{2v_i - n}\right)$$
$$= \prod_{i \neq j=1}^{n} \left(\frac{v_j - v_i - n}{v_j - v_i + n}\right) \left(\frac{v_j - v_i + 2n}{v_j - v_i - 2n}\right) \left(\frac{v_j + v_i - n}{v_j + v_i + n}\right) \left(\frac{v_j + v_i + 2n}{v_j + v_i - 2n}\right).$$

Solutions 5–9 can also be solved with the construction presented in this paper. Solution 10 as noted before is the ZF model and is solved too. The reflection matrices, $K(u)$, will be the limits of the matrices presented in section 2 and in [42, 43].

Solutions 1–4 have the problem that they represent *t–J* models and we need to use the *nested* Bethe ansatz. A possibility is to try a reformulation of the Boltzmann weights as in the cases noted in section 2 (IK model), to study the models with nested structure (*t–J* models) in a no-nests context. If this is possible we can rewrite the $R(u)$-matrix in a more appropriate form preserving the commutation relations necessary to execute the programme.

6. Conclusion

The main goal of this paper was to generalize the work of Fan in [27], based on the IK model, to all 19-vertex models. Instead, to use a particular choice of the $K(u)$-matrix as in [27] we give a formulation for all possible diagonal reflection matrices. The results were also given in a general form because it is our intention to study these models in the context of the off-shell Bethe ansatz [8], where in the semi-classical limit we have the possibility of computing exactly the n-point correlators (see, for example, [46] for the use of this approach in the $osp(1|2)$ model with periodic boundary condition).

The use of the analytical Bethe ansatz [42–44] and coordinate Bethe ansatz [47] for diagonal *K*-matrices gives many useful results about the integrable models with boundary conditions. A natural extension of these works is the possibility of studying solutions with non-diagonal *K*-matrices [48].

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Appendix. The IK model

In this appendix we will use our formulation to generate all the results presented in [27]. To obtain the results found in this appendix it is necessary to work with many trigonometric relations and it is useful to use the relations between the quotients of amplitudes found in [49] (the use of mathematical software is also recommended).

It is important to note that using the mapping presented in section 2 between the models IK and $osp(1|2)$ it is very easy to obtain the solution of the $osp(1|2)$-model from the results obtained in the appendix for the IK model.

We first need to define the amplitudes of the IK model as they appear in [27]:

$$\begin{aligned}
&x_1(u) = \sin(u+2n)\sin(u+3n) \qquad x_2(u) = -\sin(u)\sin(u+3n)\\
&x_3(u) = \sin(u)\sin(u+n)\\
&x_4(u) = -\sin(u)\sin(u+3n) + \sin(2n)\sin(3n)\\
&x_5(u) = y_5(u) = \sin(2n)\sin(u+3n)\\
&x_6(u) = -\mathrm{i}\mathrm{e}^{-\mathrm{i}2n}\sin(2n)\sin(u) \qquad y_6(u) = \mathrm{i}\mathrm{e}^{\mathrm{i}2n}\sin(2n)\sin(u)\\
&x_7(u) = \sin(u+2n)\sin(u+3n) - \mathrm{e}^{-\mathrm{i}4n}\sin(u)\sin(u+n)\\
&y_7(u) = \sin(u+2n)\sin(u+3n) - \mathrm{e}^{\mathrm{i}4n}\sin(u)\sin(u+n).
\end{aligned} \tag{84}$$

The diagonal $K(u)$-matrix has the form

$$K(u) = \mathrm{diag}\left(1, \frac{\sin\left(\frac{3}{2}n - u\right)}{\sin\left(\frac{3}{2}n + u\right)}, 1\right)$$

and the M-matrix

$$M = \begin{pmatrix} -\mathrm{i}\mathrm{e}^{\mathrm{i}4n} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\mathrm{i}\mathrm{e}^{-\mathrm{i}4n} \end{pmatrix}. \tag{85}$$

Using amplitudes (84) we can write the diagonal operators (79) as

$$\begin{aligned}
\widetilde{A}_2(u) &= A_2(u) - \frac{\sin(2n)}{\sin(2u+2n)} A_1(u)\\
\widetilde{A}_3(u) &= A_3(u) + \mathrm{i}\mathrm{e}^{4n}\frac{\sin(2n)}{\sin(2u+4n)}\widetilde{A}_2(u) - \left(1 - \mathrm{e}^{\mathrm{i}4n}\frac{\sin(2n)\sin(2u)\sin(2u+n)}{\sin(2u+2n)\sin(2u+3n)}\right) A_1(u).
\end{aligned}$$

The action of these operators in the local vacuum (80) gives us the eigenvalue

$$\begin{aligned}
\Delta_1(u) &= x_1^{2N}\\
\Delta_2(u) &= -\frac{\sin(2u)\sin\left(u+\frac{1}{2}n\right)}{\sin\left(\frac{3}{2}n+u\right)\sin(2u+2n)} x_2^{2N}\\
\Delta_3(u) &= \mathrm{e}^{\mathrm{i}4n}\frac{2\sin(2u)\cosh\left(u+\frac{5}{2}n\right)\sin\left(u+\frac{1}{2}n\right)}{\sin(2u+4n)\sin(2u+3n)} x_3^{2N}.
\end{aligned}$$

We can write the transfer matrix (81) as ($\varepsilon = 1$)

$$\tau(u) = \frac{2\cos\left(u+\frac{1}{2}n\right)\sin(2u+6n)\sin\left(u+\frac{5}{2}n\right)}{\sin(2u+2n)\sin(2u+3n)}A_1(u) + \frac{\sin(2u+6n)\sin\left(u+\frac{5}{2}n\right)}{\sin\left(u+\frac{3}{2}n\right)\sin(2u+4n)}\widetilde{A}_2(u) + \mathrm{e}^{-\mathrm{i}4n}\widetilde{A}_3(u)$$

where we use (85) to write the $K^+(u)$ in the form (7)

$$K^+(u) = \begin{pmatrix} \mathrm{i}\mathrm{e}^{\mathrm{i}4n} & 0 & 0 \\ 0 & \frac{\sinh\left(u+\frac{9}{2}n\right)}{\sinh\left(u+\frac{3}{2}n\right)} & 0 \\ 0 & 0 & \mathrm{i}\mathrm{e}^{-\mathrm{i}4n} \end{pmatrix}.$$

Inserting the amplitudes in the commuting relation for the diagonal operators we will have the relations presented in [27] (in the original paper there were only the relations for the $A_i(u)B_1(v)$). For example,

$$A_1(u)B_1(v) = \frac{\sin(u-v)\sin(u+v)}{\sin(u-v)\sin(u+v+2n)}B_1(v)A_1(u) + \frac{\sin(2n)\sin(2v)}{\sin(u-v)\sin(2v+2n)}B_1(u)A_1(v) - \frac{\sin(2n)}{\sin(u+v+2n)}B_1(u)\widetilde{A}_2(v).$$

Note that the commuting relations as presented in [27] are incomplete (the terms of type $B_i(u)C_i(v)$ vanish when applied in the local vacuum $|0\rangle$ then they were not written in the relations, but these terms are necessary for the study of two or more particle states).

The form of the eigenvalue (82) is

$$\Lambda = w_1^+\Delta_1(u)\prod_{i=1}^{n}\frac{\sin(u-v_i-2n)\sin(u+v_i)}{\sin(u-v_i)\sin(u+v_i+2n)} + w_2^+\Delta_2(u)\prod_{i=1}^{n}\frac{\sin(u-v_i-n)\sin(u-v_i+2n)\sin(u+v_i+n)\sin(u+v_i+4n)}{\sin(u-v_i)\sin(u-v_i+2n)\sin(u+v_i+3n)} + w_3^+\Delta_3(u)\prod_{i=1}^{n}\frac{\sin(u-v_i+3n)\sin(u+v_i+5n)}{\sin(u-v_i+n)\sin(u+v_i+3n)}$$

and the Bethe equations (84) have the form

$$\left(\frac{\Delta_1(v_i)}{\Delta_2(v_i)}\right)^{2N} = -\prod_{\substack{i=1\\ j\neq i}}^{n}\frac{\sin(v_j-v_i-n)\sin(v_j-v_i+2n)\sin(v_j+v_i-n)\sin(v_j+v_i+2n)}{\sin(v_j-v_i-2n)\sin(v_j-v_i+n)\sin(v_j+v_i-2n)\sin(v_j+v_i+n)} \qquad i = 1,\ldots,n$$

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